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Nonrelativistic Phase-Space and Octonions

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The octonion algebra seems to constitute the natural underlying algebra of nonrelativistic phase-space. A correspondence between the imaginary unit of quantum mechanics and the seventh imaginary unit of octonion algebra is proposed. Conjectures as to the possible physical meaning of some particular transformations from the automorphism group of octonions are presented.

1. INTRODUCTION

There are many indications that we still lack a deeper understanding of the relationship between the macroscopic continuous space of classical physics and the quantum world of elementary particles and their interactions. Among many quantum numbers characterizing elementary particles, only a few (e.g., spin, parity) have been related to space-time properties. The remaining ones (isospin, color, quark-lepton generation number) were introduced *ad hoc* in a zoological fashion when experiment forced us to accept their existence. Various proposals on how to provide them with *some kind* of geometric interpretation have been made, but none has been generally accepted.

On the other hand, the opinion that quantum theory requires a thorough revision of our standard view on the nature of space-time becomes more and more widespread. This opinion seems to be corroborated by the Bell (1964) theorem and the experiment of Aspect *et al.* (1982). The failure of naively applied classical relativistic ideas and the excellent agreement of quantum theory with the Aspect *et al.* experiment may be taken to indicate that macroscopic continuous space is a secondary concept. In fact, it has been pointed out that present theories of particle interactions should be classified as c-q theories (i.e., classical–quantum hybrids) (Finkelstein,

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1971), while the underlying theory should presumably be of a pure (and discrete) q-type (Finkelstein, 1971; Wheeler, 1973; Penrose, 1971, 1972; von Weizsäcker, 1955). According to this philosophy, the macroscopic continuous classical space and the elementary particles themselves should be constructed from an underlying quantum "pregeometry" (Wheeler, 1973; Patton and Wheeler, 1975; Penrose, 1968). Presumably, neither position nor momentum (Penrose, 1971, 1972) coordinates should appear in such an approach at the basic level. Thus, this philosophy goes further than the S-matrix approach, which dispenses with the notion of space-time continuum, but retains the continuous momentum space as one of its fundamental concepts. It is then *a priori* not excluded that within such a revised scheme it might be possible to achieve a unified (and hence geometry-*related*) interpretation of both conventional geometric and internal degrees of freedom.

Attempts to disclose the nature of quantum pregeometry centered upon the reinterpretation of quantized spin as one of its basic ingredients (Finkelstein, 1971; Penrose, 1971, 1972; Finkelstein, 1972, 1974). Therefore, if other discrete quantum characteristics of elementary particles are to be related to the properties of space in a similar manner, it seems that we should probably first seek a generalization of the classical concept of continuous rotation. A step in that direction is proposed in this paper.

The proposed generalization is based on a wish to bring more symmetry between the canonically conjugate position and momentum coordinates of nonrelativistic physics. Consideration of momentum and position coordinates (which have just been argued to be inappropriate at a deeper level) is here auxiliary and is used to arrive at a particular generalization of the concept of continuous rotation. Over 40 years ago similar symmetry arguments led Born (1949) to his reciprocity theory of elementary particles. Born's theory was based on the observation that the laws of nature (both in classical and quantum cases) seem to be invariant under four-dimensional transformations $x_k \to p_k$, $p_k \to x_k$, $k = 0, 1, 2, 3$. To allow for such transformations, he split the Planck constant into a product of fundamental length and fundamental momentum. Equivalently, we may consider the two fundamental constants to be the Planck constant (the quantum constant) and a new constant k of dimension [GeV/cm]. The introduction of k permits the combination of two separate invariants p^2 and x^2 into a single form $p^2 \pm x^2$ and the subsequent consideration of all momentum and position transformations that leave this form invariant. The reciprocity transformations of Born are among them if the "+" sign is chosen. We shall see later that the "−" sign is excluded by another argument, too.

The idea of permitting transformations of momentum into position and vice versa was considered by many physicists, but, to my knowledge, never in the form and with the interpretation suggested below. To arrive at the proposed generalization of the concept of rotation, arguments based on the properties of both classical and quantum formalisms will be employed. In the following only the six-dimensional form $\mathbf{p}^2+\mathbf{x}^2$ shall be discussed. The reasons for this restriction are as follows.

In the classical Hamiltonian formalism in which momentum and position are treated symmetrically there exists an important distinction between time (a parameter) and space (or momentum) coordinates (which are functions of this parameter). This lack of symmetry between the coordinates of space and time is even more pronounced in quantum mechanics, where time is always a parameter, while space (momentum) coordinates are operators. Serious mathematical or physical difficulties were encountered in all attempts to include time as a quantum mechanical operator [see Bayer (1983) for a review and an attempt to construct a quantum mechanical time operator].

In fact, this situation constitutes a part of the problem of a fully satisfactory unification of special relativity and quantum theory. The opinion that these theories have not been satisfactorily unified has been expressed by many physicists (Finkelstein, 1972; Dirac, 1973; Finkelstein and McCollum, 1975; Chew, 1971; Wigner, 1957). Some of the relevant arguments are as follows.

1. Present unification of the principles of relativity and quantum theory in the form of quantum field theory is formulated on the background of classical continuous space-time, which, as we have just argued, may be a bad starting point (Dirac, 1973; Finkelstein and McCollum, 1975).

2. The experimental confirmation of the existence of quantum correlations between spatially separated events raises the question of the very applicability of classical relativistic ideas to quantum systems.

3. Furthermore, in discrete finite models there exists an additional incompatibility between the principles of relativity and quantum theory: there exists no nontrivial finite-dimensional unitary representation of the Lorentz group (Finkelstein, 1972, 1974).

This situation seems to indicate that the relativistic symmetry of classical physics emerges as appropriate for the description of the world only at a higher and more complex level of the pregeometric scheme (Wheeler, 1973). In this paper we are interested in the *least possible* extension of the concept of rotation within the general $p+x$ philosophy. Therefore, the problem of how to recover relativistic invariance is left untouched.

In the next section it is shown that the symmetry group relevant for the nonrelativistic phase space is $U(1)\times SU(3)$. The $SU(3)$ symmetry group is then considered in Section 3 as the group of automorphisms of the underlying algebra. It is pointed out that the octonion algebra seems to be uniquely chosen as the natural algebra underlying phase-space transformations. The correspondence between the imaginary unit of quantum mechanics and the seventh imaginary unit of octonions is proposed there as well. Conjectures as to the possible physical meaning of some particular transformations from the automorphism group of octonions are presented in Section 4. Further possible development of the ideas of this paper is briefly touched upon in Section 5.

2. NONRELATIVISTIC PHASE SPACE

The form $\mathbf{p}^2+\mathbf{x}^2$ is invariant under the $O(6)$ group of transformations. We require that the Poisson brackets (commutators in quantum case) of the new positions and momenta $(\mathbf{p}', \mathbf{x}')$ are the same as those of the old ones $(\mathbf{p}, \mathbf{x})$, i.e., that the Poisson brackets (commutators) are form invariant:

$$\{p'_i, p'_k\} = \{p_i, p_k\} = \{x'_i, x'_k\} = \{x_i, x_k\} = 0 \tag{2.1a}$$

$$\{p'_i, x'_k\} = \{p_i, x_k\} = \delta_{ik} \tag{2.1b}$$

This requirement is weaker than that of Born, who required the invariance of the classical expression for the angular momentum as well. Since we seek a generalization of the concept of rotation, we should not require such an invariance. Equations (2.1) restrict the allowed transformations to a subset of $SO(6)$ transformations. Let us label the momentum and position coordinates as follows:

$$(z_1, z_2, z_3, z_4, z_5, z_6) = (p_1, p_2, p_3, x_1, x_2, x_3) \tag{2.2}$$

The 15 generators $G^{ik} = -G^{ki} = G^{[ik]}$ of $SO(6)$ whose defining matrix representation is

$$(G^{mn})_{ik} = \delta_i^m \delta_k^n - \delta_k^m \delta_i^n \tag{2.3}$$

satisfy the $SO(6)$ commutation relations

$$[G^{mn}, G^{kl}] = \delta^{nk} G^{ml} + \delta^{ml} G^{nk} - \delta^{mk} G^{nl} - \delta^{nl} G^{mk} \tag{2.4}$$

Since standard rotations in three-dimensional space must be understood as simultaneous rotations of p and x, we first introduce a more suitable basis for the generators of $SO(6)$;

$$\begin{gathered} J^1 = G^{32} + G^{65}, \qquad R^1 = G^{41}, \qquad H^1 = G^{62} + G^{53} \\ K^1 = G^{32} - G^{65}, \qquad Q^1 = G^{62} - G^{53} \\ \text{(and cyclically: } 1 \to 2 \to 3 \to 1,\ 4 \to 5 \to 6 \to 4) \end{gathered} \tag{2.5}$$

where J^k are the generators of standard rotations, under the operation of which the Poisson brackets (2.1) remain obviously invariant.

Let us now consider rotations in p_k-x_k planes, generated by R^k. We immediately find that under rotations by arbitrary angles in these planes the PBs of equation (2.1) still remain invariant. When enlarged by the rotations generated by $\{R^k\}$ the $so(3) \sim su(2)$ Lie algebra $\{J^k\}$ of standard rotations closes under commutation relations with the inclusion of H^k generators. Thus, the PBs (commutators in quantum case) of momentum and position are invariant under arbitrary transformations generated by the nine-dimensional Lie algebra of $\{J^k, R^k, H^k\}$.

On the other hand, a general transformation generated by K^3

$$\begin{bmatrix} p_1' \\ p_2' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \qquad p_3' = p_3$$
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_3' = x_3 \tag{2.6}$$

leaves the PBs of equation (2.1) form invariant only if $\phi = 0$, π, which, in fact, constitute particular cases of standard rotations. A similar result is obtained for Q^k.

Renaming the generators as

$$R = R^1 + R^2 + R^3$$
$$F_7 = J^1, \qquad F_5 = J^2, \qquad F_2 = -J^3, \qquad F_3 = R^1 - R^2 \tag{2.7}$$
$$F_6 = H^1, \qquad F_4 = -H^2, \qquad F_1 = -H^3, \qquad F_8 = (R^1 + R^2 - 2R^3)/\sqrt{3}$$

one calculates from (2.4) and (2.5) that (1) R commutes with all F_k, and (2) the F_k satisfy the standard commutation rules of the $su(3)$ Lie algebra

$$[F_i, F_k] = 2f_{ikj}F_j \tag{2.8}$$

with totally antisymmetric structure constants f_{ikj} equal to 1 for $ikj = (123)$, 1/2 for $ikj = (147)$, (165), (246), (257), (345), (376), $\sqrt{3}/2$ for $ikj = (458)$, (678), and zero otherwise.

The sought group of invariance is thus $U(1) \times SU(3)$. Both the reciprocity transformation $\mathbf{p}' = \mathbf{x}$, $\mathbf{x}' = -\mathbf{p}$ and the overall reflection $\mathbf{p}' = -\mathbf{p}$, $\mathbf{x}' = -\mathbf{x}$ are understood here as special kinds of *rotations in phase space* (with θ_R respectively $\pi/2$ or π) generated by the $U(1)$ generator R. The appearance of the special unitary group $SU(3)$ is well known from the nonrelativistic quantum harmonic oscillator (see, e.g., Wybourne, 1974), where it leads to additional degeneracy of the spectrum, often called "accidental" and sometimes "dynamical." Here we stress its suggested fundamental interpretation as the *minimal simple-group extension of the group of standard rotations* to those transformations of canonically conjugated momenta and positions that leave both their Poisson brackets (commutators in quantum case) and the phase-space metric $\mathbf{p}^2 + \mathbf{x}^2$ invariant.

3. OCTONIONS AS THE UNDERLYING ALGEBRA OF PHASE SPACE

It was argued that the most natural language in which both geometric (Hestenes, 1966, 1967, 1971) and pregeometric (Frescura and Hiley, 1980) concepts should be expressed is that of a "geometric algebra." For the three-dimensional classical nonrelativistic space the geometric algebra appropriate to represent rotations through algebraic multiplication was discovered 150 years ago by Hamilton (1967). Hamilton's quaternions constitute arguably the most natural language for the description of classical rotations. Multiplication of two ($k=1,2$) arbitrary elements

$$\mathbf{a}_k = a_k^0\mathbf{e}_0 + a_k^1\mathbf{e}_1 + a_k^2\mathbf{e}_2 + a_k^3\mathbf{e}_3$$

of the quaternion algebra Q is associative and distributive and is specified by the multiplication rules

$$\mathbf{e}_i\mathbf{e}_k = -\delta_{ik}\mathbf{e}_0 + \varepsilon_{ikj}\mathbf{e}_j \qquad (i, k, j = 1, 2, 3) \tag{3.1}$$

with $\mathbf{e}_0$ being the unit element of the algebra. Multiplication (3.1) may be *represented* by 2×2 Pauli matrices $(1, i * \sigma_k\}$. Coefficients a^μ are real numbers and they commute with quaternionic units $\mathbf{e}_\mu$ ($\mu = 0, 1, 2, 3$). Every quaternion possesses its conjugate $\bar{\mathbf{a}}$ obtained by reversing the signs of coefficients of $\mathbf{e}_k$ ($k = 1, 2, 3$), and norm $N(\mathbf{a})$ ($=\mathbf{a}\bar{\mathbf{a}}$), which is a real, nonnegative number.

Multiplication rules (3.1) are invariant under the $SO(3)$ group of rotations

$$\mathbf{e}'_k = O^i_k\mathbf{e}_i \tag{3.2}$$

where O belongs to $SO(3)$.

The automorphisms (3.2) may be represented by

$$\mathbf{a} \to \mathbf{a}' = \mathbf{s}\mathbf{a}\mathbf{s}^{-1} \tag{3.3a}$$

with an appropriate quaternion $\mathbf{s}$. Equation (3.3a) expresses the rotation of the scalar $a^0\mathbf{e}_0$ and the bivector $a^k\mathbf{e}_k$ in a purely quaternionic language. When further rotation described by $\mathbf{s}'$ is applied to $\mathbf{a}'$ from (3.3a), the quaternion $\mathbf{s}''$ describing rotation from $\mathbf{a}$ to $\mathbf{a}''$ is given by

$$\mathbf{s}'' = \mathbf{s}'\mathbf{s} \tag{3.3b}$$

The quaternion $\mathbf{s}$ transforming (under rotations described by $\mathbf{s}'$) according to (3.3b) is called a spinor. The geometric meaning of a spinor is operational since it describes the rotation itself: it transforms one bivector into another according to (3.3a) and one spinor into another according to (3.3b) (Hestenes, 1966, 1967, 1971).

One might consider quaternions over the field of complex numbers (a^μ complex). Such "biquaternions" are most often used in the treatment of four-dimensional relativistic space-time. In the nonrelativistic scheme the commuting "i" appears with a geometric interpretation of a unit pseudoscalar (Hestenes, 1966, 1967, 1971). However, as long as we do not deal with reflections but rotations *only*, such an additional commuting imaginary unit is superfluous (Hestenes, 1966, 1967, 1971).

In this section we shall seek a geometric algebra lying at the origin of phase-space symmetries and analogus to quaternions. Thus, the group of automorphisms of this algebra should be or should contain $SU(3)$. We should avoid "geometrically" uninterpreted "imaginary units" slipping into our construction as long as it is possible (Hestenes, 1967). Since our $SU(3)$ symmetry group does not include reflections, the commuting (pseudoscalar) "i" of Hestenes (1967) is not allowed. Thus, we restrict the field of algebra coefficients to be real (see also Jordan *et al.*, 1934). For physical reasons the sought algebra should contain the unit element to represent the identity transformation. Also, it should contain the quaternion algebra Q as a subalgebra, just as the quaternion algebra Q contains the algebra(s) of complex numbers relevant for the description of rotations in two dimensions.

The presence of the imaginary unit in the formalism of quantum mechanics was the subject of several investigations and generalizations (Jordan *et al.*, 1934., 1934; Stueckelberg, 1960; Finkelstein *et al.*, 1962, 1963). Hestenes (1966, 1967, 1971) proposed a geometric interpretation of the quantum-theoretic i in terms of the unit pseudoscalar, but this interpretation is by no means necessary (Hestenes, 1967). In fact, it is in conflict with the form invariance of the Poisson brackets (2.1b) and their quantum mechanical counterparts under ordinary reflections.

The appearance of i on the rhs of the QM version of (2.1b) strongly suggests the introduction of another imaginary unit into the underlying algebra. However, it cannot be understood as an ordinary (and commuting) i, since then the resulting algebra would be equivalent to the algebra of biquaternions, whose group of automorphisms is different from $SU(3)$. Let us call this new imaginary unit $\mathbf{e}_7$. Thus, this "i" will be treated *on equal footing* with $\mathbf{e}_k$, whose *geometrical* meaning is well appreciated.

By assumption,

$$\begin{aligned} \mathbf{e}_7\mathbf{e}_0 &= \mathbf{e}_0\mathbf{e}_7 = \mathbf{e}_7 \\ \mathbf{e}_7\mathbf{e}_7 &= -\mathbf{e}_0 \end{aligned} \tag{3.4}$$

and $\mathbf{e}_7$ transforms trivially ($\mathbf{e}_7' = \mathbf{e}_7$) under the $SO(3)$ and $SU(3)$ groups of transformations. To maintain the proposed interpretation of $\mathbf{e}_7$ we require that in subsequent multiplications of spinors (= quaternions) by $\mathbf{e}_7$ from the left the latter behaves like the ordinary imaginary unit, i.e., that after two multiplication by $\mathbf{e}_7$ a quaternion changes its sign:

$$\mathbf{e}_7(\mathbf{e}_7\mathbf{e}_k) = -\mathbf{e}_k \tag{3.5}$$

The requirement (3.5) may be rewritten with the help of the *associator*

$$(a, b, c) \equiv (ab)c - a(bc) \tag{3.6}$$

as follows:

$$(\mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_k) = 0 \tag{3.7a}$$

For multiplications of spinors by $\mathbf{e}_7$ from the right we require similarly

$$(\mathbf{e}_k, \mathbf{e}_7, \mathbf{e}_7) = 0 \tag{3.7b}$$

Let us denote

$$\mathbf{e}_{k+3} = \mathbf{e}_7\mathbf{e}_k \tag{3.8}$$

The elements $\mathbf{e}_{k+3}$ are necessarily nonzero and they constitute three additional linearly independent elements of the algebra. Clearly, $\mathbf{e}_{k+3}$ transform like $\mathbf{e}_k$ under $SO(3)$ [compare (3.2)]:

$$\mathbf{e}'_{k+3} = O^i_k \mathbf{e}_{i+3} \tag{3.9}$$

The $SO(3)$ transformations (3.2), (3.9) may be written in the form

$$\begin{bmatrix} \mathbf{e}'_1 \pm \mathbf{e}'_4 \\ \mathbf{e}'_2 \pm \mathbf{e}'_5 \\ \mathbf{e}'_3 \pm \mathbf{e}'_6 \end{bmatrix} = \begin{bmatrix} O^1_1 & O^2_1 & O^3_1 \\ O^1_2 & O^2_2 & O^3_2 \\ O^1_3 & O^2_3 & O^3_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \pm \mathbf{e}_4 \\ \mathbf{e}_2 \pm \mathbf{e}_5 \\ \mathbf{e}_3 \pm \mathbf{e}_6 \end{bmatrix} \tag{3.10a}$$

Equation (3.10a) contains two equations: either with "+" or with "−" signs.

Among the $SO(6)$ transformations on $\mathbf{e}_1, \ldots, \mathbf{e}_6$ which preserve the multiplication rule (3.8), there are simultaneous rotations in (1, 5) and (2, 4) planes. With the help of (3.5), (3.8) these may be written in a form similar to (3.10a):

$$\begin{bmatrix} \mathbf{e}'_1 \pm \mathbf{e}'_4 \\ \mathbf{e}'_2 \pm \mathbf{e}'_5 \\ \mathbf{e}'_3 \pm \mathbf{e}'_6 \end{bmatrix} = \begin{bmatrix} \cos\psi_{12} & \mathbf{e}_7 \sin\psi_{12} & 0 \\ \mathbf{e}_7 \sin\psi_{12} & \cos\psi_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \pm \mathbf{e}_4 \\ \mathbf{e}_2 \pm \mathbf{e}_5 \\ \mathbf{e}_3 \pm \mathbf{e}_6 \end{bmatrix} \tag{3.10b}$$

Two further types of such simultaneous rotations in different pairs of planes are described by formulas obtainable from (3.10b) by cyclic permutations: $1 \to 2 \to 3 \to 1$, $4 \to 5 \to 6 \to 4$.

The remaining $SO(6)$ transformations preserving (3.8) are

$$\begin{bmatrix} \mathbf{e}'_1 \pm \mathbf{e}'_4 \\ \mathbf{e}'_2 \pm \mathbf{e}'_5 \\ \mathbf{e}'_3 \pm \mathbf{e}'_6 \end{bmatrix} = \begin{bmatrix} \exp(\mathbf{e}_7\psi) & & \\ & \exp(-\mathbf{e}_7\psi) & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \pm \mathbf{e}_4 \\ \mathbf{e}_2 \pm \mathbf{e}_5 \\ \mathbf{e}_3 \pm \mathbf{e}_6 \end{bmatrix} \tag{3.10c}$$

$$\begin{bmatrix} \mathbf{e}_1' \pm \mathbf{e}_4' \\ \mathbf{e}_2' \pm \mathbf{e}_5' \\ \mathbf{e}_3' \pm \mathbf{e}_6' \end{bmatrix} = \begin{bmatrix} \exp(\mathbf{e}_7\chi) & & \\ & \exp(\mathbf{e}_7\chi) & \\ & & \exp(-2\mathbf{e}_7\chi) \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \pm \mathbf{e}_4 \\ \mathbf{e}_2 \pm \mathbf{e}_5 \\ \mathbf{e}_3 \pm \mathbf{e}_6 \end{bmatrix} \tag{3.10d}$$

$$\begin{bmatrix} \mathbf{e}_1' \pm \mathbf{e}_4' \\ \mathbf{e}_2' \pm \mathbf{e}_5' \\ \mathbf{e}_3' \pm \mathbf{e}_6' \end{bmatrix} = \exp(\mathbf{e}_7\omega) \begin{bmatrix} \mathbf{e}_1 \pm \mathbf{e}_4 \\ \mathbf{e}_2 \pm \mathbf{e}_5 \\ \mathbf{e}_3 \pm \mathbf{e}_6 \end{bmatrix} \tag{3.10e}$$

Since $(\mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_k) = (\mathbf{e}_7, \mathbf{e}_7, \mathbf{e}_{k+3}) = 0$, we may consider the 3×3 matrices in (3.10) separately from the column vectors they act on. Clearly, these matrices form the $U(1)\times SU(3)$ group.

Under the $SU(3)$ transformations of equations (3.10a)-(3.10d) the multiplication rules of our algebra are to be invariant. This requirement restricts the *a priori* arbitrary multiplication constants of $\{\mathbf{e}_n, \mathbf{e}_{m+3}\}$ as follows:

$$\begin{aligned} \mathbf{e}_m\mathbf{e}_{i+3} &= \alpha^0\delta_{mi}\mathbf{e}_0 + \alpha^7\delta_{mi}\mathbf{e}_7 - \varepsilon_{min}\mathbf{e}_{n+3} \\ \mathbf{e}_{m+3}\mathbf{e}_i &= -\alpha^0\delta_{mi}\mathbf{e}_0 - \alpha^7\delta_{mi}\mathbf{e}_7 - \varepsilon_{min}\mathbf{e}_{n+3} \\ \mathbf{e}_{m+3}\mathbf{e}_{i+3} &= -\delta_{mi}\mathbf{e}_0 - \varepsilon_{min}\mathbf{e}_n \end{aligned} \tag{3.11}$$

where $\alpha^{0,7}$ are still arbitrary real numbers. To fix them, $SU(3)$ alone is not sufficient, since it cannot transform $\mathbf{e}_7$ into any one of $\mathbf{e}_k$.

One may wonder if one could possibly impose the requirement of the invariance of the multiplication table under $U(1)$ of (3.10e). However, the $U(1)$ transformation (3.10e) leaves (3.1) invariant for $\omega = 0, \pm\frac{2}{3}\pi$ only. The remaining nontrivial discrete $U(1)$ transformations $\omega = \pm\frac{2}{3}\pi$ are equivalent to the $SU(3)$ transformation (3.10d) for $\chi = \pm\frac{2}{3}\pi$. Thus, no restrictions on $\alpha^{0,7}$ can be obtained in this way. To proceed, one has to impose some other additional assumption(s).

Before doing that, let us note that equations (3.1), (3.11) already suffice to establish [by direct calculation of the associators, say of $(\mathbf{e}_{m+3}, \mathbf{e}_i, \mathbf{e}_j)$] that the sought algebra is not associative. It may also be checked that the Jacobi identity in general is not satisfied:

$$[[\mathbf{e}_i, \mathbf{e}_j], \mathbf{e}_{m+3}] + [[\mathbf{e}_j, \mathbf{e}_{m+3}], \mathbf{e}_i] + [[\mathbf{e}_{m+3}, \mathbf{e}_i], \mathbf{e}_j] \neq 0 \tag{3.12}$$

Thus, the algebra is not Lie-admissible (Santilli, 1968; Myung, 1978).

To fix $\alpha^{0,7}$, more symmetry between $\mathbf{e}_7$ and $\mathbf{e}_k$, $\mathbf{e}_{k+3}$ is needed. Assume therefore that the following generalization of equation (3.7) holds:

$$(\mathbf{e}_M, \mathbf{e}_M, \mathbf{e}_I) = (\mathbf{e}_I, \mathbf{e}_M, \mathbf{e}_M) = 0 \qquad (M, I = 1, 2, \ldots, 7) \tag{3.13}$$

This assumption ensures that all $\mathbf{e}_M$ behave like imaginary units: in the product $\mathbf{e}_M\mathbf{e}_M(A^0\mathbf{e}_0+A^N\mathbf{e}_N)$ there is no need for parentheses specifying which multiplication should be carried out first. Assumption (3.13) suffices to fix the multiplication table completely:

$$\begin{aligned}
\mathbf{e}_m\mathbf{e}_i &= -\delta_{mi}\mathbf{e}_0+\varepsilon_{mik}\mathbf{e}_k\\
\mathbf{e}_{m+3}\mathbf{e}_{i+3} &= -\delta_{mi}\mathbf{e}_0-\varepsilon_{mik}\mathbf{e}_k\\
\mathbf{e}_m\mathbf{e}_{i+3} &= \delta_{mi}\mathbf{e}_7-\varepsilon_{mik}\mathbf{e}_{k+3}\\
\mathbf{e}_{m+3}\mathbf{e}_i &= -\delta_{mi}\mathbf{e}_7-\varepsilon_{mik}\mathbf{e}_{k+3}\\
\mathbf{e}_7\mathbf{e}_7 &= -\mathbf{e}_0\\
\mathbf{e}_7\mathbf{e}_i &= -\mathbf{e}_i\mathbf{e}_7=\mathbf{e}_{i+3}\\
\mathbf{e}_7\mathbf{e}_{i+3} &= -\mathbf{e}_{i+3}\mathbf{e}_7=-\mathbf{e}_i
\end{aligned} \tag{3.14}$$

The multiplication table (3.14) may be represented in a convenient way by the diagram of Figure 1, in which the element $\mathbf{e}_7$ has been singled out. The algebra (3.14) is the octonion division algebra Ω of Cayley (1845) and Graves (1845).

In complete analogy with quaternion conjugation, every octonion $\mathbf{A}=A^0\mathbf{e}_0+A^M\mathbf{e}_M$ possesses an octonion conjugate $\tilde{\mathbf{A}}$ obtained by replacing $\mathbf{e}_M$

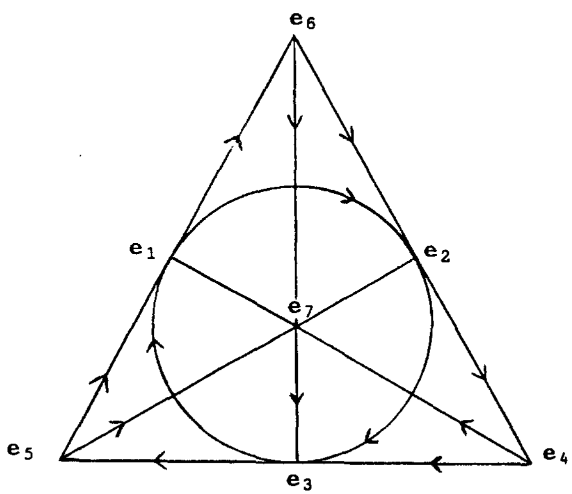

Fig. 1. Diagrammatic representation of the octonion multiplication table. Arrows indicate the directions along which multiplication has a positive sign, i.e., $\mathbf{e}_6\mathbf{e}_2=\mathbf{e}_4$, $\mathbf{e}_6\mathbf{e}_1=-\mathbf{e}_5$.

by $-\mathbf{e}_M$ $(M=1,2,\ldots,7)$, the mapping $\mathbf{A}\to\tilde{\mathbf{A}}$ being antiautomorphic: $\widetilde{(\mathbf{AB})}=\tilde{\mathbf{B}}\tilde{\mathbf{A}}$. The norm of the octonion $\mathbf{A}$ is given by

$$N(\mathbf{A})=\mathbf{A}\tilde{\mathbf{A}}=\tilde{\mathbf{A}}\mathbf{A} \tag{3.15}$$

and it defines the scalar product of two octonions $\mathbf{A}$, $\mathbf{B}$ as follows:

$$(\mathbf{A},\mathbf{B})=[N(\mathbf{A}+\mathbf{B})-N(\mathbf{A})-N(\mathbf{B})]/2 \tag{3.16}$$

Furthermore, the octonion algebra is a composition algebra, i.e., $N(\mathbf{AB})=N(\mathbf{A})N(\mathbf{B})$ and it is alternative: for any three octonions $\mathbf{A}_k$ $(k=1,2,3)$ the associator $(\mathbf{A}_i,\mathbf{A}_j,\mathbf{A}_k)$ is proportional to ε_{ijk}. Its further mathematical properties may be found in Günaydin and Gürsey (1973*a*) and Schafer (1966).

It is known that the group of automorphisms of the octonion algebra is the exceptional Lie group G_2 and that when any one of the seven imaginary units $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_7$ is held fixed the group of automorphisms is reduced to $SU(3)$ (see, e.g., Günaydin and Gürsey, 1973*a*). On the basis of the latter property we might have pointed out the octonions as a candidate underlying algebra of phase space as soon as we proposed to relate the QM i to a new $SU(3)$-invariant imaginary unit $\mathbf{e}_7$ of the algebra. We chose a different way of proceeding in order to show that the $SU(3)$ symmetry group of the nonrelativistic phase space—when combined with physically motivated requirements imposed on the sought algebra, i.e., with (1) the existence of the unit element (identity transformation), (2) the existence of quaternion subalgebra (natural description of three-dimensional rotations), (3) and the "pseudoalternativity" or "pseudodivision" property (3.7) for $\mathbf{e}_7$ (corresponding to its $i^2=-1$ interpretation)—suffice to establish that the algebra is neither associative nor Lie-admissible.

Neither the assumption that the sought algebra is a composition algebra nor the requirement of the existence of division for all elements of the algebra (which sometimes are used to single out the octonions through the Frobenius or Hurwitz theorems) played any role in our considerations. The assumption of the existence of a generalized "pseudodivision" (3.13) needed to fix all multiplication constants is still relatively weak in the sense that it admits all the hypercomplex numbers[3] formed by the Cayley-Dickson process (Dickson, 1927).

The proposal to relate the seventh octonion unit $\mathbf{e}_7$ to the standard imaginary unit i of quantum theory was made earlier in Günaydin and

[3]The "pseudodivision" property permits division for basic elements of these algebras [defined step by step in a manner analogous to (3.8)], but not for all their elements. Since the group of automorphisms of higher hypercomplex numbers is also G_2 (Schafer, 1954), it seems quite possible that they might be of some importance in the physics of phase space, too.

Gürsey (1973*b*), but (1) the relation proposed there was not as minimal as here, where the ordinary commuting i is simply absent, and (2) the scheme was not geometrically motivated.

The choice of the $\mathbf{p}^2-\mathbf{x}^2$ form would have led us along similar lines through the group $SL(3, R)$ to split octonions. Their multiplication table is obtainable from (3.14) by a formal replacement $\mathbf{e}_0 := \mathbf{e}_0, \mathbf{e}_k := \mathbf{e}_k, \mathbf{e}_{k+3} := i\mathbf{e}_{k+3}$, $\mathbf{e}_7 := i\mathbf{e}_7$, with i an ordinary complex number, playing a purely auxiliary role. We would have obtained then that $\mathbf{e}_7\mathbf{e}_7 = +\mathbf{e}_0$, in disagreement with the $i^2=-1$ property of the QM i. This QM-based argument excludes therefore the $\mathbf{p}^2-\mathbf{x}^2$ form.

It is important to realize that the G_2 group of automorphisms [although larger than $SU(3)$] does not contain the $U(1)$ factor (3.10e). Thus, the concept of three-dimensional reflections lies outside the octonion algebra.

As the quaternion algebra is the natural algebra to represent rotations in three-dimensional space, so does the octonion algebra seem to be the natural algebra underlying the transformations of the nonrelativistic phase space. It is not obvious in what way this algebra should be used in the construction of a physical theory, however. Despite that, some conjectures as to the possible physical meaning of the genuine [i.e., lying outside $SO(3)$] G_2 transformations may be offered. These conjectures are presented in the next section.

4. AUTOMORPHISMS OF OCTONIONS AND PARTICLE SYMMETRIES

For the purpose of our subsequent discussion let us first express the Dirac equation in the algebraic framework. To do this we must extend the algebra since under reflection Π the basic elements of the quaternion (the bivectors) and octonion algebras do not change: $\Pi(\mathbf{e}_M)=\mathbf{e}_M$. One achieves this extension by the introduction of a unit pseudoscalar denoted by i and subsequent direct product construction (Hestenes, 1966, 1967, 1971). The pseudoscalar i changes sign under reflections:

$$i \to \Pi(i) = -i \tag{4.1}$$

and may be represented as follows;

$$i \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -i^T, \qquad i^2=-1 \tag{4.2}$$

The operation of reflection is then represented by

$$i \to i' = PiP^{-1} = -i \tag{4.3}$$

with

$$P = \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The versors of the three-dimensional space of vectors are constructed as direct products $\boldsymbol{\alpha}_k = -i \times \mathbf{e}_k$, or

$$\boldsymbol{\alpha}_k = \begin{bmatrix} 0 & -\mathbf{e}_k \\ +\mathbf{e}_k & 0 \end{bmatrix} \tag{4.4}$$

so that $\Pi(\boldsymbol{\alpha}_k) = -PiP^{-1} \times \Pi(\mathbf{e}_k) = i \times \mathbf{e}_k = -\boldsymbol{\alpha}_k$.

The Dirac equation in momentum representation is then

$$(\alpha_k p_k + m\beta)\Psi = E\Psi \qquad \text{with} \quad \Psi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \tag{4.5}$$

and Ψ may be normalized by $\Psi^+\Psi = \bar{\Phi}_1\Phi_1 + \bar{\Phi}_2\Phi_2 = 1$. The Hermitian conjugate of the bispinor Ψ is defined here by

$$\Psi^+ \equiv \bar{\Psi}^T = [\bar{\Phi}_1, \bar{\Phi}_2] \tag{4.6}$$

(To be consistently nonrelativistic, $\mathbf{p}^2$ must be much smaller than m^2, of course.) A *very important* feature of the chosen representation is that the imaginary unit $\mathbf{e}_7$ corresponding to the customary quantum mechanical i is *not present* in the above formulas. Thus, these formulas do not contain any imaginary unit lacking a standard geometrical interpretation (see also Hestenes, 1966, 1967, 1971). Clearly, should $\mathbf{e}_7$, $\mathbf{e}_{k+3}$ appear in our formulas, the definition (4.6) of the Hermitian conjugation must be understood as containing the ($\tilde{\ }$) operation in place of ($\bar{\ }$): in general, $\Psi^+ = \tilde{\Psi}^T$.

The solution of the Dirac equation (4.5) consists in finding a pair of spinors corresponding to the given momentum and mass. Alternatively, given a pair of spinors (Φ_1, Φ_2), we may construct the corresponding momentum and mass as follows:

$$p_k\alpha_k = iE\Psi^T i\bar{\Psi} = iE(\Phi_1\bar{\Phi}_2 - \Phi_2\bar{\Phi}_1) \tag{4.7a}$$

$$m = E\Psi^T\beta\bar{\Psi} = E(\Phi_1\bar{\Phi}_2 + \Phi_2\bar{\Phi}_1) \tag{4.7b}$$

Through equation (4.7b), the mass is then expressed in terms of a product of two quaternions.

Let us now discuss briefly some particular automorphisms of the octonion algebra which lie beyond the rotation group $SO(3)$. To preserve the correspondence $\mathbf{e}_7 \leftrightarrow i_{QM}$, the G_2 transformations must be restricted to those that keep $\mathbf{e}_7$ invariant up to a sign.

4.1. Complex Conjugation

Let us denote the analog of the standard operation of complex conjugation under which $i_{QM} \to -i_{QM}$ by $(\ )^*$, i.e., $(\mathbf{e}_7)^* = -\mathbf{e}_7$. Under the operation $(\ )^*$ we have $(\mathbf{e}_k)^* = \mathbf{e}_k$ and [following definition (3.8)] $(\mathbf{e}_{k+3})^* = -\mathbf{e}_{k+3}$. The operation $(\ )^*$ is an automorphism of octonions, $(\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*$, and thus it is a G_2 transformation [obviously, however, it does not belong to our original $SU(3)$ group of transformations (3.10a)-(3.10d)]. It may be represented as a particular case ($\xi = \pi$) of the analog of (3.10c) with $\mathbf{e}_7 \to$ one of $\mathbf{e}_k$ (say, $k = 1$ and hence $\mathbf{e}'_1 = \mathbf{e}_1$), keeping also $\mathbf{e}_n$ for $n = 2, 3$ invariant:

$$\begin{bmatrix} \mathbf{e}'_4 \pm \mathbf{e}'_7 \\ \mathbf{e}'_6 \pm \mathbf{e}'_5 \\ \mathbf{e}'_2 \pm \mathbf{e}'_3 \end{bmatrix} = \begin{bmatrix} \exp(\mathbf{e}_1 \xi) & & \\ & \exp(-\mathbf{e}_1 \xi) & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_4 \pm \mathbf{e}_7 \\ \mathbf{e}_6 \pm \mathbf{e}_5 \\ \mathbf{e}_2 \pm \mathbf{e}_3 \end{bmatrix} \tag{4.8}$$

The quantum-theoretic operation of complex conjugation is known to be related to both time-reversal and charge conjugation. In standard approaches one needs *two* representations of the Lorentz group (mutually complex conjugate) to describe particles and antiparticles. The antiparticles are then interpreted as negative-energy particles moving backward in time.

The association of p_k with $\mathbf{e}_k$ in (4.7a) means that the G_2 transformation corresponding to (4.8) acts on the phase-space coordinates as follows:

$$p'_k = p_k \tag{4.9a}$$

$$x'_k = -x_k \tag{4.9b}$$

and it does not affect mass, so $m' = m$. Similarly, $E' = E$ and if $E > 0$, so is E'. The complex conjugation of the QM i ($i' = -i$) corresponding to $\mathbf{e}'_7 = -\mathbf{e}_7$ is then necessary to make QM commutation rules invariant under (4.9). If the correspondence between i_{QM} and $\mathbf{e}_7$ is to be maintained, the only reasonable interpretation of the $(\ )^*$ operation is that it leads essentially from particles to antiparticles. Particles and antiparticles should therefore be related to each other by the simultaneous operations of complex conjugation and inversion (4.9b) of the position space while keeping their momenta unchanged, (4.9a). This relationship between particles and antiparticles does not manifest itself clearly in the Dirac equation (4.5). Since the $(\ )^*$ operation is an automorphism of octonions, the transformations between particles and antiparticles lie *within* G_2.

4.2. Genuine $SU(3)$ Transformations

The quaternions $\mathbf{a} = a^0\mathbf{e}_0 + a^k\mathbf{e}_k$ form a subalgebra Q of the octonion algebra Ω. Further quaternionic subalgebras of octonions can be obtained from Q by the application of the G_2 transformations. Any such quaternionic subalgebra can be "generated" by choosing two purely imaginary orthonormal [in the sense of (3.16)] elements $\mathbf{E}_1, \mathbf{E}_2$ as two of its basis elements. The algebraic multiplication then generates the remaining element: $\mathbf{E}_3 = \mathbf{E}_1\mathbf{E}_2$ (and the unit element). For example, the original quaternion subalgebra Q is defined by choosing $\mathbf{E}_1 = \mathbf{e}_1$, $\mathbf{E}_2 = \mathbf{e}_2$ or by choosing $\mathbf{E}_1 = \mathbf{e}_2$, $\mathbf{E}_2 = \cos\zeta\, \mathbf{e}_3 + \sin\zeta\, \mathbf{e}_1$, etc.

In Sections 2 and 3 there was a complete symmetry between the coordinates of momentum and position. Such a symmetry is not present in nature, however: the four-dimensional distance x^2 is continuous, while only particular (quantized) values of p^2 are allowed for elementary particles. Thus, the introduction of the concept of mass somehow breaks the complete $p \leftrightarrow x$ symmetry. Although we do not know what the mechanism of mass generation is, it seems natural to suppose that it is related to the existence of a quaternionic subalgebra Q of octonions as exemplified by equation (4.7b) in which standard (lepton) mass is expressed solely through (pairs of) quaternions from Q. It seems then also natural to suppose that the same mass-generating mechanism works in a similar way in other isomorphic quaternionic subalgebras. Below it is pointed out that there are three such additional isomorphic subalgebras which seem to be especially interesting because they are, in a sense, maximally orthogonal to each other and to Q.

Let us consider the genuine $SU(3)$ transformations generated by F_3 [see (3.10c)]. Under these transformations the quaternion subalgebra Q is mapped onto another subalgebra $Q(F_3, \psi)$. We shall take $\psi = \pm\pi/2$. Only for these choices of ψ does there exist a possibility of choosing—as the elements generating the two subalgebras Q and $Q(F_3, \psi)$—such pairs of elements which are mutually orthogonal [i.e., $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_4, \mathbf{e}_5)$]. The resulting quaternion subalgebra $Q_3 = Q(F_3, \pm\pi/2)$ is then built upon the elements $\mathbf{e}_0, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_3$. Two further such subalgebras Q_1, Q_2 can be obtained by transformations generated by an appropriate combination of F_3 and F_8 and are built upon the elements $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_5, \mathbf{e}_6$ and $\mathbf{e}_0, \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_6$, respectively. For any two of the above four subalgebras Q_0, Q_1, Q_2, Q_3 ($Q_0 \equiv Q$) one can choose such pairs of generating elements which are mutually orthogonal. No further algebras with this property can be reached through $SU(3)$ transformations [nor through the ( )* operation]. Subalgebras Q_1, Q_2, Q_3 are obtainable from each other through ordinary three-dimensional rotations (3.10a) by $\pi/2$ in appropriate planes.

One may now immediately write down the analogs of equation (4.7) for the subalgebras Q_n:

$$\begin{aligned} p_k^{(n)}\alpha_k^{(n)} &= iE(\Phi_1^{(n)}\tilde{\Phi}_2^{(n)} - \Phi_2^{(n)}\tilde{\Phi}_1^{(n)}) \\ m^{(n)} &= E(\Phi_1^{(n)}\tilde{\Phi}_2^{(n)} + \Phi_2^{(n)}\tilde{\Phi}_1^{(n)}) \end{aligned} \tag{4.10}$$

where $\Phi_{1,2}^{(n)} \in Q_n$ and

n	$\alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)}$	$p_1^{(n)}, p_2^{(n)}, p_3^{(n)}$	$x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$
1	$-i \times \mathbf{e}_1, -i \times \mathbf{e}_5, -i \times \mathbf{e}_6$	$p_1, x_2, -x_3$	$x_1, -p_2, p_3$
2	$-i \times \mathbf{e}_4, -i \times \mathbf{e}_2, -i \times \mathbf{e}_6$	$-x_1, p_2, x_3$	$p_1, x_2, -p_3$
3	$-i \times \mathbf{e}_4, -i \times \mathbf{e}_5, -i \times \mathbf{e}_3$	$x_1, -x_2, p_3$	$-p_1, p_2, x_3$

Expressions (4.10) are not rotationally and translationally invariant. I wish to conjecture here that they should be associated with the existence of quarks and that precisely these features of theirs are responsible for the nonobservability of free quarks in our macroscopic classical world[4] to which the concepts of continuous momentum and position inherently belong. The above does not mean that in a quantum framework the quarks should not be described by Pauli spinors. However, if the above conjecture is basically correct, the use of the standard Dirac equation with its orthodox concept of quark mass is conjectured to constitute a phenomenological approximation only. This would be welcome, since in elementary particle physics the use of the Dirac equation for quarks—depending on the domain of its application—leads to two different sets of values for (so-called "current" and "constituent") quark masses (see also Schwinger, 1967; Żenczykowski, 1985).

5. OUTLOOK

Further development of the ideas of the preceding section requires presumably a thoroughly discrete (quantum) approach. In fact, we have already introduced a sort of discreteness by singling out specific discrete G_2 transformations. On the other hand, our considerations have still been classical rather than quantum. Indeed, the quaternion itself is a classical geometrical object (a combination of a scalar and a bivector). Imaginary quaternions (bivectors) describe the continuous array of rotations in the three-dimensional space. Such a continuous array of directions was shown to emerge from the spin-network combinatorial approach (Penrose, 1971, 1972). What seems to be needed is therefore such a generalization of the spin-network idea which, in the limit of large quantum numbers, would be related to octonions in a similar manner as quantum spin is related to quaternions. This hypothesized network is supposed to contain a discrete description of quark confinement and entail the emergence of qqq and $q\bar{q}$ hadrons as those objects for which the concept of mass has its orthodox meaning. To make this idea workable, a more detailed proposal concerning the mechanism of mass generation is needed. As a final remark, it should

[4]A different (geometry-unrelated) use of octonions to explain the nonobservability of quarks was proposed in Günaydin and Gürsey, 1973*c*).

be pointed out here that the above association of the network idea with the problem of strong interactions receives somewhat independent support from the success of a *W*-spin network approach to the description of hadron masses (Żenczykowski, 1988).

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